EPFL

Final Examination

MATH-337
COMBINATORIAL NUMBER THEORY
FALL SEMESTER 2023

31 JANUARY, 2024

First name: ______________________________

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Instructions

- This examination consists of 9 questions, out of which you have to answer 6. You can choose yourself which 6.
- Do not submit solutions for more than 6 questions. If you submit solutions to more, then only the ones with the least score will count towards your final grade.
- You have three hours (180 minutes) to complete this exam.
- The use of books, notes, calculators, computers, tablets or phones is prohibited.
- Write legibly and show all of your work. Unsupported answers may not earn credit. Cross out any work that you do not wish to have scored.
- Write your solutions only on the provided exam paper within the framed area. Do not use your own paper. Work written outside the margin may not be scored.
- Avoid combining solutions for multiple problems on one paper. Instead, start a new sheet for each problem's solution.
- You are permitted to use results from the course without proving them if you state and apply them correctly.
- Each question on this exam is graded out of 8 points using the same rubric as we did for homework assignments: mathematical correctness (worth 5 points) and proof-writing quality (worth 3 points). The maximal amount of points that you can score on this exam is 48.

Grading Table
(for examiner use only)

Question	Points	Score
1	8	
2	8	
3	8	
4	8	
5	8	
6	8	
7	8	
8	8	
9	8	
Total:	48	

1. (8 points) State Ramsey's Theorem for 2-sets, state Ramsey's Theorem for Graphs, and then prove that the former implies the latter.

2. (8 points) Let X be a set and let $\mathscr{P}$ be an upward closed family of subsets of X. Show that the family $\mathscr{P} \wedge \mathscr{P}^* = \{A \cap B : A \in \mathscr{P},\ B \in \mathscr{P}^*\}$ is partition regular.

3. (8 points) State and prove the Turán-Kubilius Inequality.

4. (8 points) Let $\mathscr{P} = \{A \subseteq \mathbb{N} : \forall x_1 < x_2 < \ldots \in \mathbb{N}, \exists i < j \text{ with } x_i \cdot x_j \in A\}$. Prove that $\mathscr{P}$ is closed under finite intersections.

5. (8 points) Given a set $A \subseteq \mathbb{N}$ and a natural number $b \in \mathbb{N}$, define $bA = \{bn : n \in \mathbb{N}\}$ and $A/b = \{n \in \mathbb{N} : bn \in A\}$. This problem has two parts:
   (a) Prove or provide a counterexample for the following statement: For all $b \in \mathbb{N}$ and $A \subseteq \mathbb{N}$, if A is piecewise syndetic then bA is also piecewise syndetic.
   (b) Prove or provide a counterexample for the following statement: For all $b \in \mathbb{N}$ and $A \subseteq \mathbb{N}$, if A is piecewise syndetic then A/b is also piecewise syndetic.

6. (8 points) A set $A \subseteq \mathbb{N}$ is called divisible if it contains a multiple of every natural number. Prove that there exists an ultrafilter in $\beta\mathbb{N}$ every member of which is divisible.

7. (8 points) This problem has two parts:
   (a) Let p be an ultrafilter on $\mathbb{N}$. Show that p is non-principal if and only if p contains only infinite sets.
   (b) Let p and q be ultrafilters on $\mathbb{N}$. Show that $p = q$ if and only if for all $A \in p$ and for all $B \in q$ we have $A \cap B \neq \emptyset$.

8. (8 points) Recall that a set $A \subseteq \mathbb{N}$ is called an *IP-set* if there exist $x_1 < x_2 < \ldots \in \mathbb{N}$ with $\mathrm{FS}(\{x_1, x_2, \ldots\}) \subseteq A$. Prove that if A is an IP-set then for every $m \in \mathbb{N}$ the set $\{n \in A : m \mid n\}$ is also an IP-set.

9. (8 points) Let $N \in \mathbb{N}$ and $q \in \mathbb{Z}_N \setminus \{0\}$. Given $q \in \mathbb{Z}_N \setminus \{0\}$, we call $f : \mathbb{Z}_N \to \mathbb{C}$ a *q-periodic function on $\mathbb{Z}_N$* if $f(n+q) = f(n)$ for all $n \in \mathbb{Z}_N$. Prove that a function $f : \mathbb{Z}_N \to \mathbb{C}$ is q-periodic if and only if its Fourier transform $\hat{f} : \mathbb{Z}_N \to \mathbb{C}$ satisfies $\hat{f}(\xi) \neq 0 \implies q\xi \equiv 0 \bmod N$.